崇 真 書 院

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崇真家書（36）

各位家長、各位同學：

在剛過去的星期六下午，學校按疫情指引舉辦了一個簡約的中六畢業典禮；雖然簡約，畢業生卻首次穿起崇真大使為他們設計的畢業袍，雄赳赳地接過畢業證書。在此恭喜各位畢業同學，祝各位鵬程萬里，他朝學有所成時，能回饋社會！當日我在致辭時亦說：

各位親愛的中六同學，你們是最特別的一屆，經歷了不一樣、不平凡的高中生涯——香港開埠以來沒有，環顧全球各國也沒有，像你們一樣的一屆學生。其實順境與逆境，縱橫交錯，才編織出我們與別不同的人生。「不如意事，十常八九」，與其抱怨，倒不如學習豁達面對，找出當中美善的因子，把困難看作琢磨美麗人生必要的磨刀石；那麼，逆境或許倒會擴張我們的境界。當大家日後遇到困難時，記得要保持冷靜，倚靠神，靈巧沉著應對。「你要專心仰賴耶和華，不可倚靠自己的聰明，在你一切所行的事上都要認定祂，祂必指引你的路。」（箴 3:5-6）

何謂「美善的因子」？這幾年，同學不能再完全依賴老師去學習。以往，老師東奔西跑，在課堂追功課，課後設學習小組，同學圍著老師，老師又圍著學生。老師既為師，亦似父母，諄諄善誘，千方百計幫助有需要的同學重拾學習。但疫情下，同學斷續停止面授課，即使復課，下午學校也不能舉行面授學術活動（全日復課班級例外），同學的學習成效便不得不依賴個人的自主學習能力了。「運用資訊科技進行互動學習：促進自主學習」是教育局《中學教育課程指引》（2017）的四個關鍵項目之一，文件指出學生須負起自身學習的責任，了解自己的學習需要，主動及有責任感地制定個人學習目標，篩選學習資源及決定學習策略，並能在校外使用電子學習資源進行自主學習——深度的學習。其實，學多少？學多深？從哪裡學？難道不是學習者的自由意志嗎？所謂「美善的因子」，我想正就是同學在這種環境中有關學習的體會及醒覺了，但願同學藉此疫情，能「擴張境界」吧。

上個月尾，一位中一同學因需遠赴海外升學，特來向我道別。他趁疫情期間，靠著互聯網的自學材料和老師額外的培訓，數學能力在短短時日已提升至遠超一般中六生的水平。他在離別時，留下了七頁紙的數學證明，送給他的數學老師蔡老師，蔡老師將原稿轉交學校作留念；我收到之後，欣喜非常。這是同學成長——努力學習的成果。他首先無師自通了「邏輯學」，利用反證法證明一組聯立方程的解與行列式的關係，最後自行探索行列式與向量在空間中線性相關的關係。展示的知識中，約有三成是屬於中六 M2 課程之內，七成則達至現時大學程度；而所運用的數學技巧及意境表達全屬大學水平（頭四頁如下）。

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Link between Linear Independence and Determinants

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1 Notation used

1. $M_{m\times n}(K)$ is the set of all $n \times n$ matrices with entries in K, where K is a field.
2. Let $\mathbf{v}_1, ..., \mathbf{v}_n$ be n vectors. The matrix formed by taking these n vectors and combining them, column to column, is written as: $(\mathbf{v}_1 \quad \dots \quad \mathbf{v}_n)$
3. We denote K^n for $K \times \ldots \times K$, where K is crossed with itself n times. Note that $\times$ denotes the Cartesian product.
4. We denote I_n to be the $n \times n$ identity matrix i.e. $I_n = (e_{ij})$ where $e_{ij} = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases}$.
5. Logical connectives and symbols are used in this text. Their meanings are listed below:

Symbol	Meaning
$\wedge$	and
$\vee$	or
$\sim$	not
$\Rightarrow$	therefore
$\Leftrightarrow$	if and only if

2 The proof

Before we prove the theorem linking linear independence and determinants, we will first observe and prove the following 2 lemmas and one theorem. The first theorem is listed below:

Theorem 1: Let P and Q be statements/predicates. Then, $(P \Rightarrow Q) \Leftrightarrow ((\sim Q) \Rightarrow (\sim P))$.

Proof: We will use the fact that $(P \Rightarrow Q) \Leftrightarrow ((\sim P) \vee Q)$. This is shown using a truth table:

1

P	Q	$P \Rightarrow Q$	$(\sim P) \vee Q$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	F	F

Their truth values are identical, hence the biconditional holds. Thus, we can deduce that

$$\begin{aligned} &(P \Rightarrow Q) \\ \Longleftrightarrow\ &((\sim P) \vee Q) \\ \Longleftrightarrow\ &(Q \vee (\sim P)) \\ \Longleftrightarrow\ &((\sim Q) \Rightarrow (\sim P)) \end{aligned}$$

All of these logical connectives are biconditionals. hence the final statement is a biconditional, so the theorem is proved. In particular, when the original statement itself is a biconditional, we have

$$\begin{aligned} &(P \Leftrightarrow Q) \\ \Longleftrightarrow\ &((P \Rightarrow Q) \wedge (Q \Rightarrow P)) \\ \Longleftrightarrow\ &(((\sim Q) \Rightarrow (\sim P)) \wedge ((\sim P) \Rightarrow (\sim Q))) \\ \Longleftrightarrow\ &((\sim Q) \Leftrightarrow (\sim P)) \\ \Longleftrightarrow\ &((\sim P) \Leftrightarrow (\sim Q)) \end{aligned}$$

Lemma 1: Let $A \in M_{n\times n}(K)$. There exists elementary matrices $E_1, ..., E_i$ such that $E_1...E_iA = B$, where B either is equal to I_n, or has a zero last row. In fact, $B = I_n$ if A is non-singular and B has a zero last row if A is not.

Note: When pre-multiplying a matrix by an elementary matrix, it performs an elementary row operation on that matrix.

Proof: Let $A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$. Let $\mathbf{w} = \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}$. Then, if we let $\mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ such that $A\mathbf{x} = \mathbf{w}$, we can construct a system of linear equations as follows:

$$A\mathbf{x} = \mathbf{w}$$

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}$$

2

$$\begin{pmatrix} a_{11}x_1 + \cdots + a_{1n}x_n \\ \vdots \\ a_{n1}x_1 + \cdots + a_{nn}x_n \end{pmatrix} = \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}$$

Equating coefficients, we have a system of linear equations in which the variables are $x_1, ..., x_n$. A system of linear equations can either have a unique solution, no solutions or infinitely many solutions. Elementary row operations can be used to see which case occurs. If the system has a unique solution $x_i = s_i$, we can deduce the following:

$$\begin{aligned} &\begin{cases} x_1 = s_1 \\ \vdots \\ x_n = s_n \end{cases} \\ \Longleftrightarrow\ &\begin{cases} 1x_1 + ... + 0x_n = s_1 \\ \vdots \\ 0x_n + ... + 1x_n = s_n \end{cases} \\ \Longleftrightarrow\ &\begin{pmatrix} 1x_1 + ... + 0x_n \\ \vdots \\ 0x_1 + ... + 1x_n \end{pmatrix} = \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} \\ \Longleftrightarrow\ &\begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} \\ \Longleftrightarrow\ &I_n \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} \end{aligned}$$

where $w_i = s_i$.

If the system has no or infinitely many solutions, by applying elementary row operations, the last equation becomes $0x_1 + ... + 0x_n = w_n$. The system has no (infinitely many) solutions if and only if $w_n \neq 0$ $(w_n = 0)$. Deducing further,

$$\begin{aligned} &\begin{cases} \vdots \\ 0x_1 + ... + 0x_n = w_n \end{cases} \\ \Longleftrightarrow\ &\begin{pmatrix} \vdots \\ 0x_1 + ... + 0x_n \end{pmatrix} = \begin{pmatrix} \vdots \\ w_n \end{pmatrix} \\ \Longleftrightarrow\ &\begin{pmatrix} & \vdots & \\ 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \vdots \\ w_n \end{pmatrix} \end{aligned}$$

3

We see that in both cases, the matrix A transformed to either I_n or last row zeros from elementary operations. Since we can do the same thing by repeatedly pre-multiplying A with suitable elementary matrices, the first part of the lemma follows.

By using the multilinear and alternating properties of determinants, we can deduce that the determinant of an elementary matrix is never zero. Suppose A is singular i.e. $\det A = 0$. Then, $\det(E_1...E_iA) = \det E_1...\det E_i \det A = 0$. Thus, $B \neq I_n$ since $\det I_n = 1 \neq 0$. Hence, the last row of B is zero. A similar process can be done to prove that if A is non-singular, $B = I_n$. Thus, the second part of the lemma is proven. Q.E.D.

Lemma 2: Let $A \in M_{n\times n}(K)$. Then, $\det A \neq 0$ if and only if there exists a unique solution for $\mathbf{x}$ in the equation $A\mathbf{x} = \mathbf{w}$, where $\mathbf{w}$ is an arbitrary vector in K^n.

Proof: Assume $\det A \neq 0$. Let A_i denote the matrix A whose i^{th} column is replaced by the column vector $\mathbf{w}$. Then, by Cramer's Rule, the equation has a unique solution: $x_i = \frac{\det A_i}{\det A}$, where $\mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$. Since the determinant of A is non-zero, each x_i is defined. Hence, a unique solution for $\mathbf{x}$ exists.

Assume there exists a unique vector $\mathbf{x}$ such that $A\mathbf{x} = \mathbf{w}$ no matter what $\mathbf{w}$ is. Let $\mathbf{w} = \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}$ and $\mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, where $w_i, x_i \in K$. Then, assume that $\det A = 0$. By Lemma 1, there exists elementary matrices $E_1, ..., E_i$ such that $E_1...E_iA = B$ where B has zero last row. Hence

$$A\mathbf{x} = \mathbf{w}$$

$$E_1...E_iA\mathbf{x} = E_1...E_i\mathbf{w}$$

$$B\mathbf{x} = \mathbf{a}$$

$$\begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$$

$$\begin{pmatrix} b_{11}x_1 + ... + b_{1n}x_n \\ \vdots \\ 0x_1 + ... + 0x_n \end{pmatrix} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$$

$$\Longleftrightarrow \begin{cases} b_{11}x_1 + ... + b_{1n}x_n = a_1 \\ \vdots \\ 0x_1 + ... + 0x_n = a_n \end{cases}$$

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由此可見，學習的主人是每位同學自己。學多少？有多深？真的可以自己決定！

區國年校長

二零二二年六月廿八日